

# Modular Invariants and their Fusion Rules

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*This paper is dedicated to Robert T. Powers on the occasion of his sixtieth birthday*

ABSTRACT. The subfactor approach to modular invariants gives insight into the fusion rule structure of the modular invariants.

## 1. Introduction and Background

We are going to use the inter-relations between quantum and classical dynamical systems and to use tools from non-commutative geometry or non-commutative operator algebras to understand phenomena in classical statistical mechanics.

A good starting point is the two dimensional Ising model and its C\*-treatment [1, 23, 25]. The classical model is set in the configuration space  $\mathcal{P} = \{\pm\}^{\mathbb{Z}^2}$  of distributions of + and - on the two dimensional lattice  $\mathbb{Z}^2$ , and the nearest neighbour Hamiltonian

$$H = - \sum_{\alpha, \beta \text{ } nn} J \sigma^\alpha \sigma^\beta$$

where the sum is over nearest neighbours  $\alpha$  and  $\beta$  in  $\mathbb{Z}^2$ , and  $\sigma = (\sigma^\alpha)$  in  $\mathcal{P}$ . The transfer matrix method allows us to study the classical model set in  $C(P) = \bigotimes_{\mathbb{Z}^2} \mathbb{C}^2$  and its equilibrium states, characterized by say the Dobrushin-Lanford-Ruelle equations or a variational principle by a quantum system of noncommuting observables  $\mathcal{A} = \bigotimes_{\mathbb{Z}} M_2$  in one dimension with dynamics  $\alpha_t = T^{it}(-)T^{-it}$  and associated equilibrium states or more precisely its ground states. Associated to an equilibrium state  $\mu$  at inverse temperature  $\beta$  for the Ising model is a ground state  $\varphi_\mu$  on the Pauli algebra  $\mathcal{A}$  and to each local observable  $F$  in  $C(\mathcal{P})$ , a quantum observable  $F_\beta$  depending only on the temperature such that we can describe the classical correlation values in terms of quantum ones:  $\mu(F) = \varphi_\mu(F_\beta)$ . The key

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1991 *Mathematics Subject Classification*. Primary 54C40, 14E20; Secondary 46E25, 20C20.

*Key words and phrases*. operator algebras, statistical mechanics, rational conformal field theory, modular invariants.

The second author was partially supported by FCT (Portugal) grant BD/9704/96 and CAMGSD-IST.

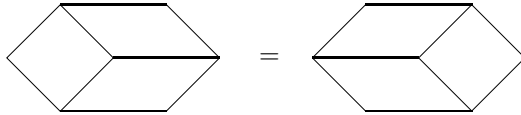


FIGURE 1. Yang-Baxter equation

element in this reduction is to identify the classical partition  $Z$  function in the denominator of the classical expectation value

$$(1.1) \quad \mu(F) = \sum_{\sigma} F(\sigma) \exp(-\beta H(\sigma)) / \sum_{\sigma} \exp(-\beta H(\sigma))$$

in quantum terms. We write for horizontal periodic boundary conditions:

$$\begin{aligned} Z &= \sum_{\sigma} \exp(-\beta H(\sigma)) \\ &= \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array} = \sum_{\sigma} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \begin{array}{|c|} \hline \\ \hline \end{array} \\ &= \sum_{\sigma} T_{\xi_1 \xi_2} T_{\xi_2 \xi_3} \dots T_{\xi_N \xi_1} = \text{trace } T \dots T = \text{trace } T^N = \text{trace } e^{-N\mathcal{H}}. \end{aligned}$$

The partition function of a square of size  $M$  by  $N$  decomposes as the exponential of a sum factorising into a product of the partition functions of columns summed over column configurations and hence can be understood as a matrix product, computed in the trace as we have imposed periodic boundary conditions. Taking the scaling limit at criticality, we obtain a field theory, and for a torus model with periodic boundary conditions in addition vertically, we have

$$(1.2) \quad Z(\tau) = \text{tr}(q^{(L_0 - c/24)} \bar{q}^{(\bar{L}_0 - c/24)}).$$

Here  $L_0$  and  $\bar{L}_0$  are commuting Hamiltonians, usually part of commuting Virasoro algebras as for the Ising model here with associated central charge  $c$ , or multiplier of the projective representation  $L_m$  of the vector fields  $l_m = -z^{m+1}d/dz$  on the circle. Much of this structure can be understood or is present in the statistical mechanical model itself, and one of our aims is to lift structures or understanding of the conformal field theories at criticality back to the original statistical mechanical models. For example, the central charge  $c$  itself can be seen from the asymptotics of the partition function  $Z$ : when  $1 \ll N \ll M$ , then  $Z \approx \exp(-NMf + M\pi c/N6)$ , where  $f$  is the free energy  $\log(Z)/MN$  after  $N, M \rightarrow \infty$  [15]. To generalise the Ising model we need some integrability of solutions of the Yang-Baxter equation as in Fig. 1. for local Boltzmann weights. Such solutions are naturally provided by Hecke algebras or quantum groups associated to  $SU(n)_k$  particularly at roots of unity - so that  $SU(2)_2$ , the Ising model is the first in a double series of examples.

The Boltzmann weights lie in

$$(\bigotimes M_n)^{SU(n)_q} = \pi(\text{Hecke}).$$

The justification of the term  $SU(n)$  models is as follows. By Weyl duality, the representation of the permutation group in  $\bigotimes M_n$  is the fixed point algebra of the product action of  $SU(n)$ . Deforming this, there is a representation of the Hecke algebra in  $\bigotimes M_n$  whose commutant is a representation of a deformation of  $SU(n)$ , the quantum group  $SU(n)_q$  [31]. The braid relations are then precisely the YBE at criticality. The representations which appear here are labelled by Young tableaux of at most  $n - 1$  rows but a further constraint implying rationality or finiteness of the representation labels occur at roots of unity  $q = e^{\pi i/(n+k)}$  when only labels with at most  $k$  columns appear. The  $SU(2)_k$  models are constructed by distributing edges of the Dynkin diagram  $A_{k+1}$  on square lattice. The first non trivial example here is described by  $A_2$ , the two extreme vertices representing our previous symbols  $\{\pm\}$  and the internal symbol is a dummy variable. The labels thus have two meanings coming from the representation theory of  $SU(n)$  of symmetric group (and their deformations) and which point of view we want to emphasise may depend on whether we are more interested in the statistical mechanics or the conformal field theory picture:

statistical mechanics	conformal field theory
Hecke algebras	$SU(N)$ loop groups
$\lambda$ symmetric or braided group representation	$\lambda$ positive energy representation of $SU(N)$
$II_1$ bimodules Jones-Wenzl subfactors	$III_1$ sectors Jones-Wassermann subfactors
Ocneanu paragroup and connections	Longo Q-systems and canonical endomorphism

## 2. Modular Invariants

The link between the two frameworks are deformed Weyl duality, Popa's classification of type III inclusion in terms of those of type II and of course the partition function  $Z$ . Decomposing Eq. (1.2) according to our underlying loop group symmetries we have:

$$Z(\tau) = \sum_{\lambda, \mu} Z_{\lambda, \mu} \chi_\lambda(\tau) \chi_\mu(\tau)^*,$$

where  $\chi_\lambda(\tau) = \text{tr}_{H_\lambda} \exp(2\pi i \tau L_0)$  is the character in the positive energy representation  $\lambda$  and  $q = e^{2\pi i \tau}$ . Modular invariance under reparameterisation of the torus then becomes  $Z(\tau) = Z((a\tau + b)/(c\tau + d))$  for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  in  $SL(2; \mathbb{Z})$  which is generated

by  $\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Since  $SL(2; \mathbb{Z})$  acts linearly on characters, there is a representation of  $SL(2; \mathbb{Z})$  by  $\mathcal{S} \mapsto S = [S_{\lambda, \mu}]$ , and  $\mathcal{T} \mapsto T = [T_{\lambda, \mu}]$ , where

$$\chi_{\lambda}(-1/\tau) = \sum_{\mu} S_{\lambda, \mu} \chi_{\mu}(\tau), \quad \chi_{\lambda}(\tau + 1) = \sum_{\mu} T_{\lambda, \mu} \chi_{\mu}(\tau).$$

Then modular invariance is

$$(2.1) \quad ZS = SZ, ZT = TZ.$$

Since the coefficients  $Z_{\lambda\mu}$  appeared as multiplicities,

$$(2.2) \quad Z_{\lambda\mu} \in 0, 1, 2, \dots,$$

and usually through uniqueness of the vacuum we have

$$(2.3) \quad Z_{00} = 1.$$

A modular invariant is then a matrix  $Z = [Z_{\lambda\mu}]$  satisfying the above constraints Eq. (2.1), Eq. (2.2) and Eq. (2.3), although we will see that much interesting structure is uncovered if we relax the normalization condition  $Z_{00} = 1$ . Usually as in the  $SU(n)_k$  case, the modular data is such that  $S$  is symmetric and  $S_{0\lambda} > 0$ .

In this case [9, 28]

$$(2.4) \quad 0 \leq Z_{\lambda, \mu} \leq d_{\lambda} d_{\mu}$$

where  $d_{\lambda} = S_{0, \lambda}/S_{00}$ , and thus there are at most finitely many normalized modular invariants.

There always exists the trivial modular invariant  $Z = 1$ . For  $SU(2)$  there is the celebrated A-D-E classification [14], e.g.  $SU(2)_{16}$  have three modular invariants labelled by the Dynkin diagrams  $A_{17}, D_{10}, E_7$  with corresponding invariants  $Z_{A_{17}} = 1$ ,  $Z_{D_{10}}$  is an orbifold obtained by folding the  $A_{17}$ -diagram which has a fixed point, and  $Z_{E_7}$  is a twist of the orbifold invariant. These are the normalized modular invariants, and if one relaxes the normalization condition,  $Z_{00} = 1$ , to consider for example products of normalized modular invariants then an interesting fusion rule algebra appears as in Table 1.

	$Z_{A_{17}}$	$Z_{D_{10}}$	$Z_{E_7}$
$Z_{A_{17}}$	$Z_{A_{17}}$	$Z_{D_{10}}$	$Z_{E_7}$
$Z_{D_{10}}$	$Z_{D_{10}}$	$2Z_{D_{10}}$	$2Z_{E_7}$
$Z_{E_7}$	$Z_{E_7}$	$2Z_{E_7}$	$Z_{D_{10}} + Z_{E_7}$

TABLE 1. Fusion of  $SU(2)_{16}$  modular invariants

It is then natural to ask whether the (unnormalized) modular invariants form a fusion rule algebra generated by the normalized modular invariants and get a better understanding of these fusion rule algebras. The subfactor approach can assist us in this [21, 26].

### 3. Moore Seiberg dilation

Here we will have a type III factor  $N$ , endowed with a system  $\mathcal{A}$  of endomorphisms which are taken to be braided and so yield a modular data. As we have seen we always have the trivial modular invariant

$$(3.1) \quad Z = \sum_{\lambda} |\chi_{\lambda}|^2,$$

where we now interpret  $\chi_\lambda$  as a formal character  $\text{tr} q^{L_0 - c/24}$  even when the Hamiltonian  $L_0$  may not exist. More generally we may introduce twists or permutations of the fusion rules preserving  $S$  and  $T$  and the vacuum  $0$  so that we should also consider

$$(3.2) \quad Z = \sum_\lambda \chi_\lambda \chi_{\vartheta(\lambda)}^*.$$

In some sense every modular invariant can be dilated or brought to this form. Suppose we can extend the system in the following sense. We have a subfactor  $N \subset M$  with a system  $\mathcal{B}$  of endomorphisms of  $M$ . One can emphasise the extension aspect of moving from endomorphisms of  $N$  to those of  $M$  ( $\alpha$ -induction [34, 6]). One needs the system  $\mathcal{A}$  to be braided and the sector of  $\bar{u}$ , the dual canonical endomorphism to lie in  $\Sigma(\mathcal{A})$ , the set of finite integral sums of endomorphism of  $\mathcal{A}$ .

Under these conditions we emphasise the restriction aspect of this extension. The trivial or twisted modular invariants Eq. (3.2) for the  $\mathcal{B}$ -system can be restricted to  $\mathcal{A}$ , written formally in terms of characters as  $\chi_\tau = \sum_\lambda b_{\tau,\lambda} \chi_\lambda$ , with branching coefficients  $b_{\tau,\lambda}$ . Then restricting the diagonal modular invariant

$$(3.3) \quad Z^{\text{ext}} = \sum_{\tau \in \mathcal{B}} |\chi_\tau|^2,$$

to the original system we have:

$$(3.4) \quad \sum_{\tau \in \mathcal{B}} |\chi_\tau|^2 = \sum_{\tau \in \mathcal{B}} \left| \sum_{\lambda \in \mathcal{A}} b_{\tau,\lambda} \chi_\lambda \right|^2,$$

with mass matrix

$$(3.5) \quad Z_{\lambda,\mu} = \sum_\tau b_{\tau,\lambda} b_{\tau,\mu}.$$

These invariants are called type I (or more precisely the inclusion  $N \subset M$  describing this modular invariant is type I) and are necessarily symmetric  $Z_{\lambda\mu} = Z_{\mu\lambda}$ . In the presence of non-trivial twist  $\vartheta$  of the  $\mathcal{B}$  system, we have type II invariants

$$(3.6) \quad Z_{\lambda,\mu} = \sum_\tau b_{\tau,\lambda} b_{\vartheta(\tau),\mu}.$$

which have symmetric vacuum coupling  $Z_{0\lambda} = Z_{\lambda 0}$ .

A type III phenomena occurs, due to some underlying heterotic structure which results in needing different labellings  $\mathcal{B}_+$  and  $\mathcal{B}_-$  on left and right extended systems  $\mathcal{A} \subset \mathcal{B}_\pm$ . In the subfactor framework, this is found in two intermediate subfactors  $N \subset M_\pm \subset M$ , where  $M_\pm$  carry systems of endomorphisms  $\mathcal{B}_\pm$ . Not only can this situation be found with modular invariants with non-symmetric vacuum coupling as in the orthogonal at loop groups low levels [8], but also with quantum doubles of finite groups [26]. Turning to the extension point of view  $N \subset M$ , when  $\mathcal{A} \subset \text{End}(N)$  is non-degenerately braided and the dual canonical endomorphism  $\theta$  lies in  $\Sigma(\mathcal{A})$ , we form  $\mathcal{C}^\pm$  to be the systems of endomorphisms on  $M$  from the irreducible components of the sectors of  $\{\alpha_\lambda^\pm : \lambda \in \mathcal{A}\}$ . These generate the full system  $\mathcal{C} = \mathcal{C}^+ \vee \mathcal{C}^-$ , not necessarily even commutative, but  $\mathcal{B} = \mathcal{C}^0 = \mathcal{C}^+ \cap \mathcal{C}^-$  is not only commutative but non-degenerately braided [8].

#### 4. Izumi quantum $E_6$ model

Non-degenerately braided systems and the corresponding modular data can be obtained in the operator algebra setting in at least two ways. One is to take the Jones-Wassermann loop group examples in algebraic quantum field theory analysed by Wasserman [43] and students e.g. Laredo-Toledano [41]. The other is to take the quantum double of a system of endomorphisms - which may not even be commutative let alone braided or may be braided but the braiding has some degeneracy. Here we focus on one such example the quantum double of the even  $E_6$  system. (As explained in [13, Section 5], this is more natural than taking the double of the entire  $E_6$  system as the Longo-Rehren construction then gives the double of the  $E_6$  subfactor). The  $E_6$  system can be realised through a conformal embedding of  $SU(2)_{10} \subset SO(5)_1$  [44, 6, 7]. The modular data obtained from the Longo-Rehren inclusion of the even system of  $E_6$  is explicitly written down as follows in [30, Page 648] where  $d = 1 + \sqrt{3}$ ,  $\lambda = 2 + d^2$ ,  $i = \sqrt{-1}$ . The  $S$  and  $T$  matrices are

$$S = \frac{1}{\lambda} \begin{pmatrix} 1 & 1 & 1+d & 1+d & 2+d & d & d & d & d & d \\ 1 & 1 & 1+d & 1+d & -2-d & d & d & -d & -d & -d \\ 1+d & 1+d & 1 & 1 & 2+d & -d & -d & -d & -d & -d \\ 1+d & 1+d & 1 & 1 & -2-d & -d & -d & d & d & d \\ 2+d & -2-d & 2+d & -2-d & 0 & 0 & 0 & 0 & 0 & 0 \\ d & d & -d & -d & 0 & d & d & -d & -d & 2d \\ d & d & -d & -d & 0 & d & d & -d & -d & -2d \\ d & -d & -d & d & 0 & -d & d & -2i-di & 2i+di & 0 \\ d & -d & -d & d & 0 & -d & d & 2i+di & -2i-di & 0 \\ d & -d & -d & d & 0 & 2d & -2d & 0 & 0 & 0 \end{pmatrix}$$

$$T = \text{diag}(1, -1, 1, -1, 1, e^{\pi i/3}, e^{4\pi i/3}, e^{5\pi i/6}, e^{5\pi i/6}, -i).$$

**4.1. Verlinde matrices of the quantum  $E_6$  model.** We computed numerically the Verlinde matrices of the quantum double of  $E_6$ :  $N_0$  is the identity matrix, and the others can be written as the following quadratic expressions

$$\begin{aligned} N_1 &= \chi_0\chi_1^* + \chi_1\chi_0^* + \chi_2\chi_3^* + \chi_3\chi_2^* + |\chi_4|^2 + \chi_5\chi_6^* + \chi_6\chi_5^* + \chi_7\chi_8^* + \chi_8\chi_7^* + |\chi_9|^2, \\ N_2 &= \chi_0\chi_2^* + \chi_2\chi_0^* + \chi_1\chi_3^* + \chi_3\chi_1^* + |\chi_2 + \chi_3|^2 + (\chi_2 + \chi_3)(\chi_5 + \chi_6)^* \\ &\quad + (\chi_5 + \chi_6)(\chi_2 + \chi_3)^* + 2|\chi_4|^2 + \chi_4(\chi_7 + \chi_8 + \chi_9)^* + (\chi_7 + \chi_8 + \chi_9)\chi_4^* \\ &\quad + \chi_5\chi_6^* + \chi_6\chi_5^* + \chi_7(\chi_8 + \chi_9)^* + (\chi_8 + \chi_9)\chi_7^* + \chi_8\chi_9^* + \chi_9\chi_8^*, \\ N_3 &= \chi_0\chi_3^* + \chi_3\chi_0^* + \chi_1\chi_2^* + \chi_2\chi_1^* + |\chi_2 + \chi_3|^2 + (\chi_2 + \chi_3)(\chi_5 + \chi_6)^* \\ &\quad + (\chi_5 + \chi_6)(\chi_2 + \chi_3)^* + 2|\chi_4|^2 + \chi_4(\chi_7 + \chi_8 + \chi_9)^* + (\chi_7 + \chi_8 + \chi_9)\chi_4^* \\ &\quad + |\chi_5|^2 + |\chi_6|^2 + |\chi_7|^2 + |\chi_8|^2 + \chi_9(\chi_7 + \chi_8)^* + (\chi_7 + \chi_8)\chi_9^*, \\ N_4 &= (\chi_0 + \chi_1 + 2\chi_2 + 2\chi_3)\chi_4^* + \chi_4(\chi_0 + \chi_1 + 2\chi_2 + 2\chi_3)^* + \chi_4(\chi_5 + \chi_6)^* \\ &\quad + (\chi_5 + \chi_6)\chi_4^* + (\chi_2 + \chi_3 + \chi_5 + \chi_6)(\chi_7 + \chi_8 + \chi_9)^* \\ &\quad + (\chi_7 + \chi_8 + \chi_9)(\chi_2 + \chi_3 + \chi_5 + \chi_6)^*, \\ N_5 &= (\chi_0 + \chi_3 + \chi_5)\chi_5^* + \chi_5(\chi_0 + \chi_3)^* + |\chi_2 + \chi_3|^2 \\ &\quad + (\chi_4 + \chi_7 + \chi_8 + \chi_9)\chi_4^* + \chi_4(\chi_7 + \chi_8 + \chi_9)^* + (\chi_1 + \chi_2 + \chi_6)\chi_6^* \\ &\quad + \chi_6(\chi_1 + \chi_2 + \chi_6)^* + \chi_7\chi_8^* + \chi_8\chi_7^* + |\chi_9|^2, \end{aligned}$$

$$\begin{aligned}
N_6 &= \chi_0\chi_6^* + \chi_6\chi_0^* + \chi_1\chi_5^* + \chi_5\chi_1^* + |\chi_2 + \chi_3|^2 + \chi_2\chi_5^* + \chi_5\chi_2^* + \chi_3\chi_6^* \\
&\quad + \chi_6\chi_3^* + (\chi_4 + \chi_7 + \chi_8 + \chi_9)\chi_4^* + \chi_4(\chi_7 + \chi_8 + \chi_9)^* + \chi_5\chi_6^* \\
&\quad + \chi_6\chi_5^* + |\chi_7|^2 + |\chi_8|^2 + |\chi_9|^2, \\
N_7 &= \chi_0\chi_7^* + \chi_8\chi_0^* + \chi_1\chi_8^* + \chi_7\chi_1^* + \chi_2(\chi_4 + \chi_8 + \chi_9)^* + (\chi_4 + \chi_7 + \chi_9)\chi_2^* \\
&\quad + \chi_3(\chi_4 + \chi_7 + \chi_9)^* + (\chi_4 + \chi_8 + \chi_9)\chi_3^* + (\chi_5 + \chi_6)\chi_4^* + \chi_4(\chi_5 + \chi_6)^* \\
&\quad + \chi_5\chi_8^* + \chi_7\chi_5^* + \chi_6\chi_7^* + \chi_8\chi_6^*, \\
N_8 &= \chi_0\chi_8^* + \chi_7\chi_0^* + \chi_1\chi_7^* + \chi_8\chi_1^* + \chi_2(\chi_4 + \chi_7 + \chi_9)^* \\
&\quad + (\chi_4 + \chi_8 + \chi_9)\chi_2^* + (\chi_4 + \chi_7 + \chi_9)\chi_3^* + \chi_3(\chi_4 + \chi_8 + \chi_9)^* \\
&\quad + (\chi_5 + \chi_6)\chi_4^* + \chi_4(\chi_5 + \chi_6)^* + \chi_5\chi_7^* + \chi_7\chi_6^* + \chi_6\chi_8^* + \chi_8\chi_5^*, \\
N_9 &= \chi_0\chi_9^* + \chi_9\chi_0^* + (\chi_2 + \chi_3)(\chi_4 + \chi_7 + \chi_8)^* + (\chi_4 + \chi_7 + \chi_8)(\chi_2 + \chi_3)^* \\
&\quad + (\chi_5 + \chi_6)(\chi_4 + \chi_9)^* + (\chi_4 + \chi_9)(\chi_5 + \chi_6)^* + \chi_1\chi_9^* + \chi_9\chi_1^*.
\end{aligned}$$

All the  $N_i$ 's are symmetric apart from  $N_7$  and  $N_8$  that are transpose of each other. The first two matrices  $N_0$  and  $N_1$  are permutation matrices as the primary fields 0 and 1 are the simple currents of our present modular data. The quantum  $E_6$  modular invariant  $Z_2$  is the charge conjugation invariant, i.e.  $[Z_2]_{\lambda, \bar{\mu}} = \delta_{\lambda, \bar{\mu}}$ . The Frobenius-Schur indicator  $\text{FS}_\lambda = 1$  for all  $\lambda \in \mathcal{A}$  except for  $\lambda = 7, 8$  where it is zero.

**4.2. Modular invariants of the quantum  $E_6$  subfactor(s).** The dimension of the commutant  $\{S, T\}'$  is four and spanned by the modular invariants which are exactly four and were computed numerically using [10, Eq. (1.3)]. We obtain  $Z_1 = N_0$ , and the others are given by the following quadratic expressions:

$$\begin{aligned}
Z_2 &= |\chi_0|^2 + |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + |\chi_6|^2 + |\chi_9|^2 + \chi_7\chi_8^* + \chi_8\chi_7^*, \\
Z_3 &= |\chi_0 + \chi_2|^2 + |\chi_1 + \chi_3|^2 + 2|\chi_4|^2, \\
Z_4 &= |\chi_0 + \chi_2 + \chi_4|^2.
\end{aligned}$$

**4.3. Subfactors for quantum  $E_6$  modular invariants.** Here we consider the Longo-Rehren  $N \subset M$  inclusion of the even system of the  $E_6$  Dynkin diagram, and let  $\mathcal{A}$  be the system of endomorphisms that yield the above modular data. By [26, Lemma 3.8] (see also [40]) we can choose an endomorphism  $\lambda_1$  from the sector  $[\lambda_1]$  such that  $\lambda_1^2 = 1$  since  $T_{\lambda_1, \lambda_1} = 1$ . Therefore  $[\theta] = [\lambda_0] \oplus [\lambda_1]$  is a dual sector of the braided subfactor  $N \subset N \rtimes \mathbb{Z}_2$ . The irreducible decomposition of the  $N$ - $M$  sectors is as follows:  $[\iota\lambda_0] = [\iota\lambda_1]$ ,  $[\iota\lambda_2] = [\iota\lambda_3]$ ,  $[\iota\lambda_4] = [a_1] \oplus [a_2]$ ,  $[\iota\lambda_5] = [\iota\lambda_6]$ ,  $[\iota\lambda_7] = [\iota\lambda_8]$ , and  $[\iota\lambda_9] = [a_3] \oplus [a_4]$ . Hence gives rise to a trace 8 modular invariant. So  $Z_{N \subset N \rtimes \mathbb{Z}_2} = Z_2$  the permutation invariant. Second the  $\text{LR}(E_6)$  dual canonical sector is  $[\theta] = [\lambda_0] \oplus [\lambda_2] \oplus [\lambda_4]$ . The irreducible decomposition of the  $N$ - $M$  sectors are as follows:  $[\iota\lambda_0], [\iota\lambda_1], [\iota\lambda_2] = [\iota\lambda_0] \oplus [\iota\lambda_5]$ ,  $[\iota\lambda_3] = [\iota\lambda_1] \oplus [\iota\lambda_5]$ ,  $[\iota\lambda_4] = [\iota\lambda_0] \oplus [\iota\lambda_1] \oplus [\iota\lambda_5]$ ,  $[\iota\lambda_5] = [\iota\lambda_6] = [\iota\lambda_7] = [\iota\lambda_8] = [\iota\lambda_9]$ . Therefore  $\text{LR}(E_6)$  produces a trace 3 modular invariant. Hence  $Z_{\text{LR}(E_6)} = Z_4$ . By [13, Sect. 5] the notation of Fig. 1 on page 28,  $[(0, 0)] \oplus [(6, 0)]$  is the  $[\lambda_0] \oplus [\lambda_2]$  in our current notation. In [13, Example 5.1] instead of taking the three even vertices of  $E_6$  one can take the two extreme endpoints (which are even and discard the internal point) following Izumi's Galois correspondence [29, Proposition 2.4]. In this way, we get an intermediate subfactor  $N \subset P \subset M$  with inclusion map  $\iota_P$  such that  $[\theta] = [\lambda_0] \oplus [\lambda_2]$  is the sector of the dual canonical endomorphism of  $N \subset P$ . Then by the Verlinde matrices we find that:  $[\iota_P\lambda_0], [\iota_P\lambda_1], [\iota_P\lambda_2] = [\iota_P\lambda_0] \oplus [\iota_P\lambda_5]$ ,

$[\iota_P \lambda_4] = [\iota_P \lambda_9] \oplus [a] \oplus [b]$ ,  $[\iota_P \lambda_5] = [\iota_P \lambda_6]$ ,  $[\iota_P \lambda_7] = [\iota_P \lambda_8]$ ,  $[\iota_P \lambda_9]$ , which gives trace 6 modular invariant. Thus the subfactor  $N \subset P$  produces  $Z_3$ . Moreover, chiral locality holds true here. We thus have proven the following.

**THEOREM 4.1.** *All the modular invariants of the quantum double of the  $E_6$  system are realised by braided subfactors.*

The fusion rules between the quantum  $E_6$  double (sufferable) modular invariants are as in Table 2. It is worth mentioning that as the modular invariants form a basis of the commutant  $\{S, T\}'$  the coefficients exist and are uniquely determined, but due to a fusion rule algebra phenomenon those coefficients are non-negative integers [26, Corollary 3.6] (see also [21]).

	$Z_1$	$Z_2$	$Z_3$	$Z_4$
$Z_1$	$Z_1$	$Z_2$	$Z_3$	$Z_4$
$Z_2$	$Z_2$	$Z_1$	$Z_3$	$Z_4$
$Z_3$	$Z_3$	$Z_3$	$2Z_3$	$2Z_4$
$Z_4$	$Z_4$	$Z_4$	$2Z_3$	$3Z_4$

TABLE 2. Fusion  $Z_a Z_b^t$  of modular invariants

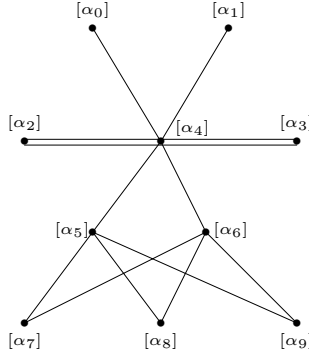


FIGURE 2.  $Z_1$ , fusion graph of  $[\alpha_4^+]$

**4.4. Full systems of the quantum  $E_6$  modular invariants.** The global index of our system is  $\omega = 8(1 + d + d^2) \sim 89.5692$ .

*Case  $Z_1$ .* For the trivial modular invariant  $Z_1$  we have  $\mathcal{A} \simeq \mathcal{C}^0 \simeq \mathcal{C}$ . We display the fusion graph of  $[\alpha_4]$  in Fig. 2.

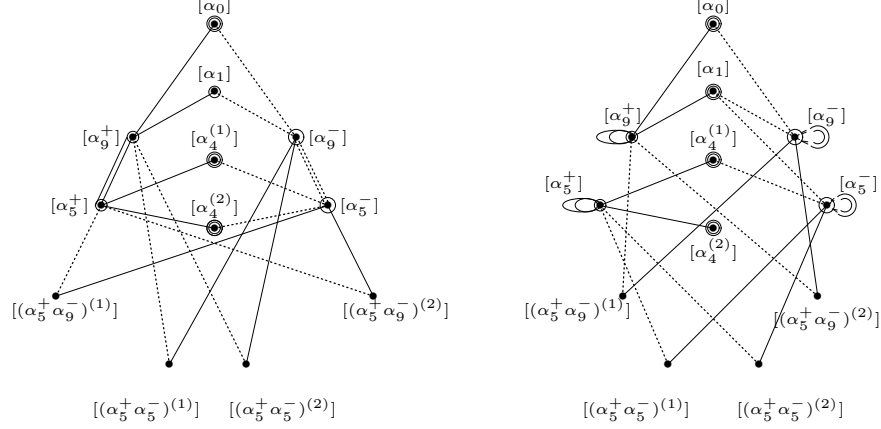
*Case  $Z_2$ .* For the permutation invariant  $Z_2$ , the full system  $\mathcal{C}$  is obtained by permuting the sectors of those in the case  $Z_1$ .

*Case  $Z_3$ .* For the sufferable quantum  $E_6$  modular invariant  $Z_3$  we have by [12]:

$$\omega_{\pm} = 8(1 + d + d^2)/(2 + d) \sim 18.9282, \quad \omega_0 = 4.$$

Computing using the Verlinde matrices we find:  $\langle \alpha_0, \alpha_0 \rangle = \langle \alpha_1^{\pm}, \alpha_1^{\pm} \rangle = \langle \alpha_5^{\pm}, \alpha_5^{\pm} \rangle = \langle \alpha_6^{\pm}, \alpha_6^{\pm} \rangle = \langle \alpha_7^{\pm}, \alpha_7^{\pm} \rangle = \langle \alpha_8^{\pm}, \alpha_8^{\pm} \rangle = 1$ ,  $\langle \alpha_2^{\pm}, \alpha_2^{\pm} \rangle = \langle \alpha_3^{\pm}, \alpha_3^{\pm} \rangle = 2$ ,  $\langle \alpha_4^{\pm}, \alpha_4^{\pm} \rangle = 3$ ,  $\langle \alpha_0, \alpha_i^{\pm} \rangle = \delta_{0,i} + \delta_{2,i}$ ,  $\langle \alpha_1^{\pm}, \alpha_i^{\pm} \rangle = \delta_{1,i} + \delta_{3,i}$ ,  $\langle \alpha_2^{\pm}, \alpha_3^{\pm} \rangle = \langle \alpha_2^{\pm}, \alpha_5^{\pm} \rangle = \langle \alpha_2^{\pm}, \alpha_6^{\pm} \rangle = \langle \alpha_3^{\pm}, \alpha_5^{\pm} \rangle = \langle \alpha_3^{\pm}, \alpha_6^{\pm} \rangle = 1$ ,  $\langle \alpha_4^{\pm}, \alpha_7^{\pm} \rangle = \langle \alpha_4^{\pm}, \alpha_8^{\pm} \rangle = \langle \alpha_4^{\pm}, \alpha_9^{\pm} \rangle = 1$ ,  $\langle \alpha_5^{\pm}, \alpha_6^{\pm} \rangle = \langle \alpha_7^{\pm}, \alpha_8^{\pm} \rangle = \langle \alpha_7^{\pm}, \alpha_9^{\pm} \rangle = \langle \alpha_8^{\pm}, \alpha_9^{\pm} \rangle = 1$ , and the others vanish. We then conclude



FIGURE 3.  $Z_3$ , fusion graphs of  $[\alpha_9^\pm]$  and  $[\alpha_5^\pm]$  where  $Z_3^2 = 2Z_3$ 

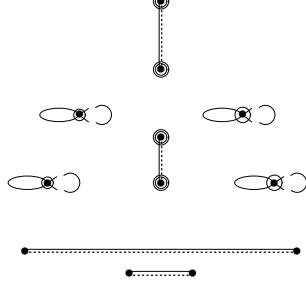
that  $[\alpha_0]$  and  $[\alpha_1^\pm]$  are irreducible, and  $[\alpha_2^\pm] = [\alpha_0] \oplus [\alpha_5^\pm]$ ,  $[\alpha_3^\pm] = [\alpha_1^\pm] \oplus [\alpha_5^\pm]$ ,  $[\alpha_4^\pm] = [\alpha_9^\pm] \oplus [\alpha_4^{\pm(1)}] \oplus [\alpha_4^{\pm(2)}]$ ,  $[\alpha_5^\pm] = [\alpha_6^\pm]$ ,  $[\alpha_7^\pm] = [\alpha_8^\pm] = [\alpha_9^\pm]$ . Using also the matrix  $Z=Z_3$  and noting that  $[\alpha_4^{\pm(i)}] = [\alpha_4^{\mp(i)}]$  for  $i = 1, 2$  we conclude that  $\mathcal{C}^0 = \{\alpha_0, \alpha_1, \alpha_4^{(1)}, \alpha_4^{(2)}\}$ , and  $\mathcal{C}^\pm = \mathcal{C}^0 \cup \{\alpha_5^\pm, \alpha_9^\pm\}$ . Moreover since

$$\langle \alpha_4^+ \alpha_1^-, \alpha_4^+ \rangle = N_{4,4}^\xi Z_{1,\xi} = 3 = \langle \alpha_4^+ \alpha_1^-, \alpha_4^+ \alpha_1^- \rangle$$

we have the sectors of  $\mathcal{C}^0$  equal  $\mathbb{Z}_2 \times \mathbb{Z}_2$  with  $[\alpha_4^{(i)}][\alpha_4^{(i)}] = [\text{id}]$ . We need to find four more irreducible sectors in the full system  $\mathcal{C}$  because  $\#\mathcal{C} = \text{Tr}(ZZ^t) = 12$ . We compute and find that

$$\langle \alpha_5^+ \alpha_5^-, \alpha_5^+ \alpha_5^- \rangle = \sum N_{5,5}^\xi N_{5,5}^\eta Z_{\xi,\eta} = 2$$

and the dimensions of the intertwiner spaces between  $\alpha_5^+ \alpha_5^-$  and any sector in both chiral systems  $\mathcal{C}^\pm$  vanish. Hence  $[\alpha_5^+ \alpha_5^-]$  decomposes into two new irreducible sectors  $[(\alpha_5^+ \alpha_5^-)^{(1)}]$  and  $[(\alpha_5^+ \alpha_5^-)^{(2)}]$ . The other two sectors are similarly obtained from the decomposition of  $\alpha_9^+ \alpha_5^-$  further noting that  $\langle \alpha_5^+ \alpha_5^-, \alpha_9^+ \alpha_5^- \rangle = 0$ . We denote these new irreducible sectors by  $[(\alpha_9^+ \alpha_5^-)^{(1)}]$  and  $[(\alpha_9^+ \alpha_5^-)^{(2)}]$ . We have the fusion graphs of both  $[\alpha_9^+]$  and  $[\alpha_9^-]$  displayed on the LHS of Fig. 3. We use straight lines for the fusion graph of  $[\alpha_9^+]$  and dashed lines for that of  $[\alpha_9^-]$ . On the RHS of Fig. 3 the fusion graph of  $[\alpha_5^+]$  and  $[\alpha_5^-]$  is displayed. We display the fusion graph of  $[\alpha_1^+]$  and  $[\alpha_1^-]$  in Fig. 4. When  $\theta$  is the local dual canonical endomorphism  $\lambda_0 \oplus \lambda_2$ , let  $\gamma$  denote the canonical sector of  $N \subset M$ . Since  $\langle \alpha_5^+ \alpha_5^-, \gamma \rangle = \langle \alpha_5^+, \alpha_5^+ \rangle = 1$  we conclude from [7, Corollary 3.19] that  $[\gamma] = [\alpha_0] \oplus [(\alpha_5^+ \alpha_5^-)^{(1)}]$ . The full system  $\mathcal{C}$  decomposes as two sheets. The first is the chiral system  $\mathcal{C}^+$  and the second sheet comprises the irreducible components of  $\mathcal{C}^+ \alpha_5^-$  in accordance with the decomposition of  $Z_3^2 = 2Z_3$  and  $[\gamma] = [\alpha_0] \oplus [(\alpha_5^+ \alpha_5^-)^{(1)}]$  and [26, Corollary 3.6] and [21].

FIGURE 4.  $Z_3$ , fusion graph of  $[\alpha_1^\pm]$  where  $Z_3^2 = 2Z_3$ 

Case  $Z_4$ . The global indices of the chiral systems  $\mathcal{C}^\pm$  and  $\mathcal{C}^0$  are by [12]:

$$\omega_\pm = 8(1 + d + d^2)/(4 + 2d) = 2 + d^2 \sim 9.4641, \quad \omega_0 = \omega_\pm^2/\omega = 1$$

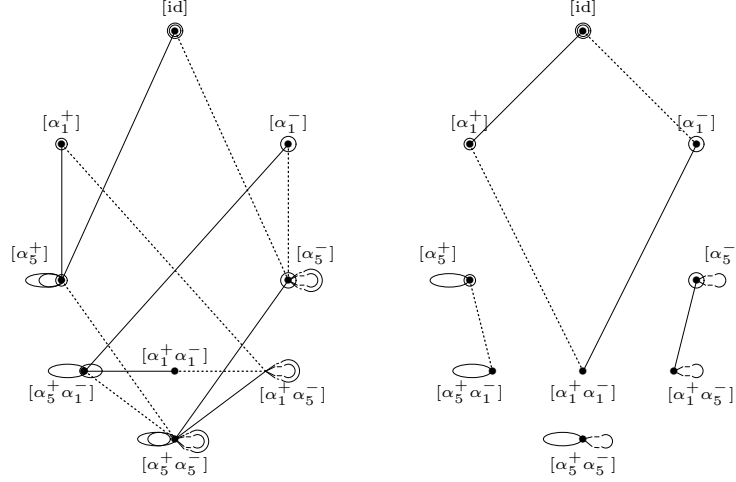
respectively. There are  $9 = \text{Tr}(Z_4 Z_4^t)$   $M$ - $M$  irreducible sectors in the full commutative chiral system  $\mathcal{C}$ . Note that  $\mathcal{C}^0 = \{\alpha_0\}$  since  $\omega_0 = 1$ . We next compute the chiral systems  $\mathcal{C}^\pm$ . We get (using the streamlined notation  $\alpha_{\lambda_i}^\pm =: \alpha_i^\pm$ ):  $\langle \alpha_0^\pm, \alpha_0^\pm \rangle = \langle \alpha_1^\pm, \alpha_1^\pm \rangle = \langle \alpha_4^\pm, \alpha_4^\pm \rangle = \langle \alpha_5^\pm, \alpha_5^\pm \rangle = \langle \alpha_6^\pm, \alpha_6^\pm \rangle = \langle \alpha_7^\pm, \alpha_7^\pm \rangle = \langle \alpha_8^\pm, \alpha_8^\pm \rangle = \langle \alpha_9^\pm, \alpha_9^\pm \rangle = 1$ ,  $\langle \alpha_2^\pm, \alpha_2^\pm \rangle = \langle \alpha_3^\pm, \alpha_3^\pm \rangle = 2$ ,  $\langle \alpha_4^\pm, \alpha_4^\pm \rangle = 3$ ,  $\langle \alpha_0^\pm, \alpha_i^\pm \rangle = \delta_{0,i} + \delta_{2,i} + \delta_{4,i}$ ,  $\langle \alpha_1^\pm, \alpha_i^\pm \rangle = \delta_{1,i} + \delta_{3,i} + \delta_{4,i}$ ,  $\langle \alpha_2^\pm, \alpha_3^\pm \rangle = \langle \alpha_2^\pm, \alpha_5^\pm \rangle = 1$ ,  $\langle \alpha_2^\pm, \alpha_4^\pm \rangle = 2$ ,  $\langle \alpha_3^\pm, \alpha_4^\pm \rangle = 2$ ,  $\langle \alpha_3^\pm, \alpha_5^\pm \rangle = \langle \alpha_3^\pm, \alpha_6^\pm \rangle = 1$ ,  $\langle \alpha_4^\pm, \alpha_5^\pm \rangle = \langle \alpha_4^\pm, \alpha_6^\pm \rangle = \langle \alpha_5^\pm, \alpha_6^\pm \rangle = \langle \alpha_5^\pm, \alpha_7^\pm \rangle = \langle \alpha_5^\pm, \alpha_8^\pm \rangle = \langle \alpha_5^\pm, \alpha_9^\pm \rangle = 1$ .

Therefore  $\mathcal{C}^\pm = \{\alpha_0, \alpha_1^\pm, \alpha_5^\pm\}$  and as sectors  $[\alpha_2^\pm] = [\alpha_0] \oplus [\alpha_5^\pm]$ ,  $[\alpha_3^\pm] = [\alpha_1^\pm] \oplus [\alpha_5^\pm]$ ,  $[\alpha_4^\pm] = [\alpha_0] \oplus [\alpha_1^\pm] \oplus [\alpha_5^\pm]$ ,  $[\alpha_5^\pm] = [\alpha_6^\pm] = [\alpha_7^\pm] = [\alpha_8^\pm] = [\alpha_9^\pm]$ .

The fusion rules are as the well known fusion rules of the even vertices of the graph  $E_6$  [13]:  $[\alpha_5^\pm \alpha_5^\pm] = [\alpha_0] \oplus [\alpha_1^\pm] \oplus 2[\alpha_5^\pm]$ ,  $[\alpha_1^\pm][\alpha_5^\pm] = [\alpha_5^\pm]$ ,  $[\alpha_1^\pm][\alpha_1^\pm] = [\alpha_0]$ . We find that  $\mathcal{C} = \mathcal{C}^+ \times \mathcal{C}^-$  and remark that  $d_{\alpha_5^\pm} = 1 + \sqrt{3}$ . We display the fusion the all  $M$ - $M$  system together with the fusion graphs of both  $[\alpha_5^-]$  and  $[\alpha_5^+]$  in the LHS Fig. 5. In this figure, we use straight lines for the fusion graph of  $[\alpha_5^+]$  whereas dashed lines for that of  $[\alpha_5^-]$ . We also encircled the  $\mathcal{C}^+$ -chiral sectors with small circles and with larger circles those for the  $\mathcal{C}^-$ -chiral system. The RHS of Fig. 5 we display the fusion graphs of  $[\alpha_1^\pm]$ . If  $\theta$  is the local dual canonical endomorphism  $\lambda_0 \oplus \lambda_2 \oplus \lambda_4$ , let  $\gamma$  be the corresponding canonical endomorphism of  $N \subset M$ . As an application of [7, Corollary 3.19], we can compute that  $[\gamma] = [\alpha_0] \oplus [\alpha_1^+ \alpha_1^-] \oplus [\alpha_5^+ \alpha_5^-]$ . In accordance with  $Z_4^2 = 3Z_4$  and [26, Corollary 3.6] the full system  $\mathcal{C}$  decomposes as three sheets  $\mathcal{C}^+$ ,  $\mathcal{C}^+ \alpha_1^-$  and  $\mathcal{C}^+ \alpha_5^-$ .

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FIGURE 5.  $Z_4$ , fusion graphs of  $[\alpha_5^\pm]$  and  $[\alpha_1^\pm]$  where  $Z_4^2 = 3Z_4$ 

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